International Journal of Science & Technology ISSN (online): 2250-141X
www.ijst.co.in Vol. 1 Issue 1, November 2011

Numerical Solution of Boundary Layer Flow past a stretching plate

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Abstract

Many problems in science and engineering can be reduced to the problem of solving differential equation satisfying given certain conditions. The analytical method of solution, with which the reader is assumed to be familiar, can be applied to solve only a selected class of differential equations. Those equations which govern physical system do not possess, in general closed form solution, and hence recourse must be made to numerical methods for solving such differential equations.

There are well developed techniques to solve by analytical methods, certain particular types of differential equations. However a vase majority of differential equation are such that their closed form solutions can not be obtained using the available analytical methods.

In such situations several useful numerical methods are available to solve the equations. Even those equations which have analytical solutions, can be solved numerically with greater case. A comparison of numerical solution with the corresponding analytical solution will show how close the former is to the later.

In the present work we consider a linear stretching plate whose velocity is proportional to the distance from the slit. A boundary layer how of construct properties. We neglect any disturbance of flow created by the slit. The sheet has infinite width, we have analytical solution for velocity field. We apply the similarity transformation to reduce the problem to boundary value problem of ordinary differential equation. The obtained ordinary differential equation is nonlinear boundary value problem, there is no prescribed method. So, mostly nonlinear are dealt with numerical techniques.

In the present problem, we reduce the nonlinear boundary value problem to nonlinear initial value problem by applying Shooting technique. Later we employ Runga-Kutta method to solve initial value problem.

Nomenclature

u velocity component in x-direction;
v velocity component in y-direction;
ν Kinematics viscocity of air;
y distance in y-direction of x-axis;

ISSN (online): 2250-141X
Vol. 1 Issue 1, November 2011

1.1 Introduction

The flow past a stretching plate has great importance in many industrial applications such as in polymer industry to draw plastic films and artificial fibers. In the process of artificial fibers the polymer emerges from an orifice with speed, which increases from zero at the orifice up to a plateau value at which it remains constant. The moving fiber produces a boundary layer in the medium.Surrounding the fiber, which is of technical importance in that it governs the rate at which the fibers is cooled and this is in turn affect, the final properties of the yarn. Crane[3] investigated boundary layer from past a stretching sheet whose velocity is proportional to distance from the slit. Carragher[4] reconsidered the problem of Crane[3,4] to study transfer and calculated the Nusselt number for the entire range of prandtl number. Prandtl, Crane and Carragher both considered the stretching plate without pores.

In the present study, we consider a linear stretching plate whose velocity is proportional to the distance from the slit and boundary layer flow of constant properties neglecting any disturbance of flow created by the slit and the sheet has infinite width Crane[3] obtained analytic solution for velocity field. We apply the similarity transformation to reduce the problem to boundary value problem of ordinary differential equation. The obtained ordinary differential equation is nonlinear of order three. To solve nonlinear boundary value problem, there is no prescribe method. So mostly nonlinear problem are dealt with numerical techniques.

In the present problem, we reduce the non linear boundary value problem to non linear initial value problem by applying shooting technique. Later we apply Runge-Kutta method to solve initial value problem. We compare our result, with the result obtained by Naseem Ahmad[2].

BOUNDARY VALUE PROBLEM (Two point boundary value problem)

A differential equation of order two

$$y'' + a_1(x)y' + a_2(x)y = r(x)$$

Together with

$$y(x_0) = \alpha_0, y(x_n) = \alpha_1$$

Is called boundary value problem. The conditions $y(x_0) = \alpha_0, y(x_n) = \alpha_1$ are called boundary conditions.

Regarding the solution of boundary value problems, we have no result to guaranty the solution and uniqueness as in the case of initial value problems. To be sure about the solution of boundary value problem, we be comfortable by recasting the boundary value problem. In turn, we apply Runge-Kutta method to solving reduced boundary value problem. To change boundary value problem into initial value problem w apply shooting methods.

ISSN (online): 2250-141X
Vol. 1 Issue 1, November 2011

SHOOTING METHOD:

This method required good initial guesses for the slope and can be applied to both linear and n on-linear problems. Its main advantage is that it is easy to apply. We discuss this method with reference to the second order boundary problem defined by

$$y''(x) = f(x) \qquad (1)$$
$$y(0) = 0, y(1) = 1$$

The main steps involved in this method are

(i) Transformation of the boundary value problem into an initial value problem,

(ii) Solution of the initial value problem by Taylor's series or Runga-Kutta method etc, and finally,

(iii) Solution of the given boundary value problem.

To apply any initial method, we must know $y'(0)$. Let us assume that the true value of $y'(0)$ is m. We start with two initial guesses for m and then determine the corresponding values of $y(1)$ using any initial value method. Let the two guesses be m0 and m1 and also let the corresponding values of $y(1)$, obtained by the initial value method , be denoted by $y(m_0;1)$ and $y(m_1;1)$, respectively. Using linear interpolation, we then obtain a better approximation m2 for m. This is given by

$$\frac{m_2 - m_0}{y(1) - y(m_0;1)} = \frac{m_1 - m_0}{y(m_1;1) - y(m_0;1)} \qquad (2)$$

Which gives

$$\frac{y(1) - y(m_0;1)}{y(m_1;1) - y(m_0;1)} \qquad (3)$$

We now solve the initial value problem

$$y''(x) = f(x) \qquad (4)$$
$$y(0) = 0,\ y'(0) = m_2$$

And obtain $y(m_2;1)$. We again use linear interpolation with $\{m_1, y(m_1;1)\}$ and $\{m_2, y(m_2;1)\}$ to obtain a better approximation m_3 for m and so on. The process is repeated until convergence is obtained, i.e, until the value of $y(m_1;1)$ agrees with $y(1)$ to the desired accuracy. The speed of convergence depends, of course, on how good the initial guesses were. The method will be tedious to apply to higher order boundary value problems and in the case of non-linear problem, linear interpolation yields unsatisfactory results.

Real problems cannot be restricted to linear second order ordinary differential equations. In the most of cases, the governing differential equation comes out to be non-linear of order more than two. Therefore, it is essential that we must have the method, which can be applied for solving the differential equation of order more than two. We are developing Runga-Kutta method for third order differential equation.

ISSN (online): 2250-141X
Vol. 1 Issue 1, November 2011

We consider the following differential equation of order third

$y''' = f(x, y, y', y'')$ (1) With the initial condition

$y(x_0) = y_0, y'(x_0) = y'_0, y''(x_0) = y''_0$ (2)

By setting

$$y' = \frac{dy}{dx} = u(x, y, y', y'')$$

$$y'' = \frac{du}{dx} = v(x, y, y', y'') \qquad (3)$$

$$y''' = \frac{dv}{dx} = w(x, y, y', y'')$$

With the initial conditions

$y = y_0, y' = y'_0, y'' = y''_0$ when $x = x_0$.

Assuming that $\Delta x = h, \Delta y = k, \Delta y' = l$ and $\Delta y'' = p$, the fourth order Runga-Kutta method gives

$$k_1 = hu(x_0, y_0, y'_0, y''_0)$$

$$l_1 = hv(x_0, y_0, y'_0, y''_0)$$

$$p_1 = hw(x_0, y_0, y'_0, y''_0)$$

$$k_2 = hu\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, y'_0 + \frac{l_1}{2}, y''_0 + \frac{p_1}{2} \right)$$

$$l_2 = hv\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, y'_0 + \frac{l_1}{2}, y''_0 + \frac{p_1}{2} \right)$$

$$p_2 = hu\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, y'_0 + \frac{l_1}{2}, y''_0 + \frac{p_1}{2} \right)$$

RUNGA-KUTTA METHOD FOR THIRD ORDER DIFFERENTIAL EQUATION:

$$k_3 = hu\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, y'_0 + \frac{l_2}{2}, y''_0 + \frac{p_2}{2} \right)$$

$$l_3 = hv\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, y'_0 + \frac{l_2}{2}, y''_0 + \frac{p_2}{2} \right)$$

$$p_3 = hu\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, y'_0 + \frac{l_2}{2}, y''_0 + \frac{p_2}{2} \right)$$

$$k_4 = hu(x_0 + h, y_0 + k_3, y'_0 + l_3, y''_0 + p_3)$$

$$l_4 = hv(x_0 + h, y_0 + k_3, y'_0 + l_3, y''_0 + p_3)$$

$$p_4 = hu(x_0 + h, y_0 + k_3, y'_0 + l_3, y''_0 + p_3)$$

$$\therefore x_1 = x_0 + h$$

$$y = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y' = y'_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$y'' = y''_0 + \frac{1}{6}(p_1 + 2p_2 + 2p_3 + p_4)$$

Hence, continuing the same process we get the numerical solution given by the following table.

S.No	x	y	y'	y''
1	x_0	y_0	y'_0	y''_0
2	x_1	y_1	y'_1	y''_1
3	x_2	y_2	y'_2	y''_2
4	x_3	y_3	y'_3	y''_3
-	-	-	-	-
-	-	-	-	-
n	x_n	y_n	y'_n	y''_n

ISSN (online): 2250-141X
Vol. 1 Issue 1, November 2011

1.2 FORMATION OF PROBLEM

Let the x-axis taken along the flow and y-axis normal to it. If u and v are the velocity component along these directions, respectively, then under the usual boundary layer approximation, the steady flow problem is governed by the equation

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (1)$$

Momentum equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \qquad (2)$$

Where ν is the kinematics viscosity.The relevant boundary conditions are

$$y=0,\quad u=mx,\quad v=-v_0,\quad m>0,\quad v_0>0$$
$$y=\infty,\quad u=0,\quad v=f(\infty)$$

1.2 SOLUTION OF THE PROBLEM

Introducing dimensionless variables

$$y'=\frac{y}{h},\quad u'=\frac{uh}{\nu},\quad x'=\frac{x}{h},\quad v'=\frac{vh}{\nu}$$

Equation (1) and (2) reduces to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (3)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \qquad (4)$$

With the boundary conditions

$$\begin{matrix} y=0, & u=x \\ y=\infty, & u=0 \end{matrix} \qquad (5)$$

We set the similarity solution of the forms

$$u=mxf'(y) \qquad (6)$$

Substituting equation (6) in the equation (3) and applying condition (5), we have

$$v=-\{f(y)-f(0) \qquad (7)$$

Using u and v in the equation (4), we have

$$f^{iv}(y)-f(y)f''(y)=f'''(y) \qquad (8)$$

With boundary conditions

$$\begin{matrix} y=0, & f=0, & f'=1 \\ y=\infty, & f'=0, & c=f(\infty) \end{matrix} \qquad (9)$$

Where $f(0)=0,$ with loss of generality

Equation (8) is nonlinear boundary value problem of order three. This boundary value problem can not be solved numerically because the range is infinite. Therefore, for physical justification, we may redefine the problem as

$$f'''(y)=f'^2(y)-f(y)f''(y) \qquad (10)$$

Together with the boundary conditions

$$\begin{matrix} y=0, & f=0, & f'=1 \\ y=\infty, & f'=0, & c=f(1) \end{matrix} \qquad (11)$$

This problem has an unique solution if it is redefined as initial value problem. So

ISSN (online): 2250-141X
Vol. 1 Issue 1, November 2011

S.N	Iteration	Approximate Values of f''	Error
1.	m_0	0.6	1.83
2.	m_1	0.7	1.93
3.	m_2	1.2297	.0003
4.	m_3	1.2308	.000

Hence $f''(0) = -1.23$, therefore, our initial value problem to be solved is

$f'''(y) = f'^2(y) - f(y)f''(y)$ (12)

With initial conditions

$y = 0, \quad f = 0, \quad f' = 1, \quad f'' = -1.23$

Now we have to solve the initial value problem (12) of order three by using Runga-Kutta method of order four to split the initial value problem (12) into three linear differential equations of order one as follows. Let

$f'(y) = u(y, f, f', f'')$ (13)

With the initial conditions

$y = 0, \quad f = 0, \quad f' = 1, \quad f'' = -1.23$

$u'(y) = v(y, f, f', f'')$ (14)

With the initial conditions

$y = 0, \quad f = 0, \quad f' = 1, \quad f'' = -1.23$

And $v'(y) = w(y, f, f', f'')$ (15)

With the initial conditions

$y = 0, \quad f = 0, \quad f' = 1, \quad f'' = -1.23$

Now, we applying the Runga-Kutta method to solve the above equations.

Now, first we take

$h = 0.1, \quad y_0 = 0, \quad f_0 = 0, \quad f_0' = 1, \quad f_0'' = -1.23$

applying shooting method and reduce it into initial value problem as follows

Table 1.1

$K_1 = hu(y_0, f_0, f_0', f_0'') = 0.1 \times u(0,0,1,-1.23)$
$= 0.1$

$L_1 = hv(y_0, f_0, f_0', f_0'') = 0.1 \times v(0,0,1,-1.23)$
$= -0.123$

$P_1 = hw(y_0, f_0, f_0', f_0'') = 0.1 \times w(0,0,1,-1.23)$
$= 0.1$

$$K_2 = hu\left(y_0 + \frac{h}{2}, f_0 + \frac{K_1}{2}, f_0' + \frac{L_1}{2}, f_0'' + \frac{P_1}{2} \right)$$

$$= 0.1 \times u(0.05, 0.05, 0.9385, -1.18) = 0.0938$$

$$L_2 = hv\left(y_0 + \frac{h}{2}, f_0 + \frac{K_1}{2}, f_0' + \frac{L_1}{2}, f_0'' + \frac{P_1}{2} \right)$$

$$= 0.1 \times v(0.05, 0.05, 0.9385, -1.18) = -0.118$$

$$P_2 = hw\left(y_0 + \frac{h}{2}, f_0 + \frac{K_1}{2}, f_0' + \frac{L_1}{2}, f_0'' + \frac{P_1}{2} \right)$$

$$= 0.1 \times w(0.05, 0.05, 0.9385, -1.18) = 0.094$$

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Hence after solving the differential equations (13) (14) and (15) numerically by Runga-Kutta method of order four, we get the following results in table 1.2.

Table 1.2

ISSN (online): 2250-141X
Vol. 1 Issue 1, November 2011

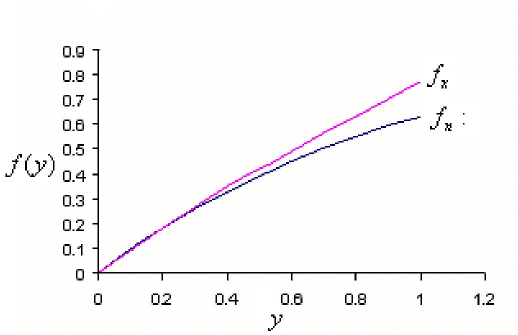
1.4 DISCUSSION AND RESULTS

We have computed the velocity function $f(y)$ and its derivatives $f'(y)$, $f''(y)$ By Runga-Kutta method of fourth order. Naseem Ahmad et al [2], obtained the expression for $f(y)$ and $f'(y)$ by similarity transformation as

$$f(y) = 1 - e^{-y}$$

and $f'(y) = e^{-y}$

We are comparing our computed results for $f(y)$ and $f'(y)$ with exact solution obtained by Naseem Ahmad et al [2]. On seeing the graph of computed values of exact solutions, we have the following conclusions



f_p : Stands for present numerical solution.

f_n : Stands for solution obtained by N. Ahmad et al.

Figure 1.1 shows that the computed results shown by a curve f_p and the

Figure 1.1, Plot of $f(y)$ verses y.

S.No	y	f	f'	f''
1.	0	0	1	-1.23
2.	0.1	.094	.8818	-1.1359
3.	0.2	.1766	.7724	-1.0527
4.	0.3	.2487	.6709	-.9797
5.	0.4	.3110	.5763	-.9136
6.	0.5	.3642	.4879	-.8554
7.	0.6	.4088	.4045	-.8033
8.	0.7	.4453	.3265	-.7566
9.	0.8	.4742	.2530	-.7143
10.	0.9	.4964	.1841	-.6760
11.	1.00	.5111	.1183	-.6406

curve of exact solution given by f_n are almost coinciding for $0 \le y \le 0.5$ and afterward f_p deviates from exact solution and we agree in
the accuracy of the order of 10^{-1}. Hence the computed function is agreeing within the boundary layer and the computed function deviates from function as we move away the boundary layer.

ISSN (online): 2250-141X
Vol. 1 Issue 1, November 2011

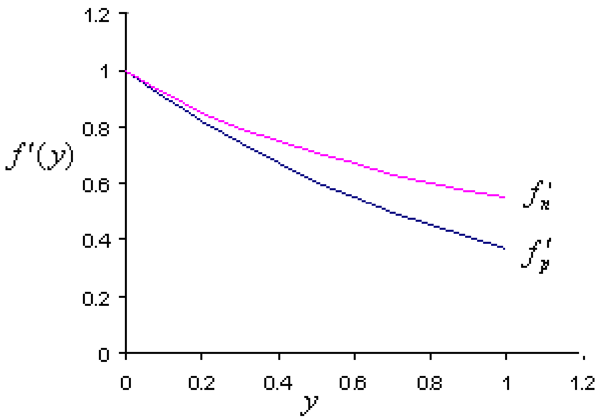


Figure1.2 Plot of $f'(y)$ verses y.

f'_p: Stands for present numerical solution.

f'_n: Stands for solution obtained by Naseem Ahmad et al.

Figure 1.2 exhibits the comparison of computed values of $f'(y)$ shown the curve f'_p while f'_n is the graph of $f'(y) = e^{-y}$. We again observe that f'_p and f'_n agree for $y \in [0, 0.5]$ and f'_p deviate from f'_n when we move from the stretching plate.

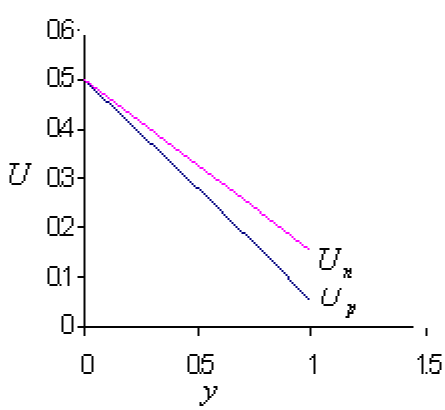


Figure 1.3 Plot of U verses y

U_p: stands for present Numerical solution.

U_n: stands for solution obtained by N. Ahmad et al.

In figure 1.3, we see that for some particular value of x taken as x=0.5, the numerical solution for velocity component v is not in good agreement. This Numerical method is valid only in immediate neighborhood of stretching plate. We deviate from exact solution as we move away from the plate.

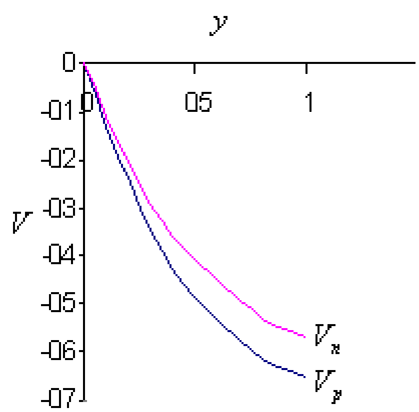


Figure 1.4 plot of V verses y

V_p: stands for present Numerical solution.

V_n: stands for solution obtained by N. Ahmad et al.

In figure 1.4, we notice that computed velocity V_p and the velocity V_n are agreed in good approximation. Both the velocities V_p and V_n are almost same for and $y \in [0, 0.5]$ later V_p deviates from exact solution V_n. Hence, our numerical solution is well accurate in boundary layer while we lose the accuracy as we move away the stretching plate.

ISSN (online): 2250-141X
Vol. 1 Issue 1, November 2011

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