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INTEGRAL SOLUTIONS OF TERNARY QUADRATIC DIOPHANTINE EQUATION $x^2 + 7y^2 = 16z^2$

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ABSTRACT

The ternary homogenous quadratic Diophantine equation representing cone given by $x^2 + 7y^2 = 16z^2$ is analyzed for finding its non-zero distinct integral solutions. Four different patterns of integer solutions are presented. A few interesting relations between the solutions and special number patterns are given.

Keywords: *Ternary quadratic, homogenous quadratic, integer solutions, special number 2010 Mathematics Subject Classification: 11D09*

1. INTRODUCTION

The Ternary Quadratic Diophantine Equation offers an unlimited field for research because of their variety [1-2]. For an extensive review of various problems, one may refer [3-5]. This communication concerns with yet another interesting Ternary Quadratic equation

$x^2 + 7y^2 = 16z^2$ representing a homogenous cone for determining its infinitely may non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

Notations used:

$T_{m,n}$= Polygonal number of rank n with sides m.

2. Method of Analysis

The ternary Quadratic equation to be solved for its non-zero integer solution is

$$x^2 + 7y^2 = 16z^2 \qquad (1)$$

We present below different patterns of non-zero distinct integer solutions to (1)

Pattern : 1

Write (1) as

$(x + 3z)(x - 3z) = 7(z + y)(z - y)$,

which is written in the form of ratio as

$$\frac{(x+3z)}{(z+y)} = 7\frac{(z-y)}{(x-3z)} = \frac{A}{B}, B \neq 0 \qquad (2)$$

This is equivalent to the following two equations

$Bx - Ay - (A - 3B)z = 0$

$-Ax - 7By + (3A + 7B)z = 0$

Applying the method of cross multiplication, we get

$$\left.\begin{aligned} x &= x(A, B) = -3A^2 - 14AB + 21B^2 \\ y &= y(A, B) = A^2 - 6AB - 7B^2 \\ z &= z(A, B) = -A^2 - 7B^2 \end{aligned}\right\} \qquad (3)$$

Thus, (3) represents non-zero distinct integral solutions of (1) in two parameters.

Properties:-

1. $3y(A, A+1) + x(A, A+1) = -32 Pr_A$

2. $x(A, 1) + 3t_{4,A} + G_{7A} \equiv 0 \pmod 2$

3. $x(1, B) - 21t_{4,B} + G_{7B} \equiv 0 \pmod 2$

4. $3y(1, B) - x(1, B) + 42t_{4,B} - G_{2B} \equiv 1 \pmod 3$

5. $y(1, B) + 7t_{4,B} + 2G_{3B} \equiv 0 \pmod 2$

6. $z(1, B) + 7t_{4,B} + 1 = 0$.

7. $y(2, B) + 7t_{4,B} + 2G_{6B} \equiv 1 \pmod 2$

8. $4x(1,B) + 2z(1, B) - 70 t_{4,B} + 2 G_{28} \equiv 1 \pmod 2$

Pattern: 2

Introducing the transformations

$$x = 3\alpha,\ y = x + 16T,\ z = x + 7T \qquad (4)$$

in (1), we have

$$\alpha^2 = x^2 - 112T^2, \qquad (5)$$

which is satisfied by

$$\left.\begin{aligned} T &= T(A, B) = 2AB \\ x &= x(A, B) = 112A^2 + B^2 \\ \alpha &= \alpha(A, B) = 112A^2 - B^2 \end{aligned}\right\} \qquad (6)$$

Substituting (6) in (4), the non – zero distinct integral solutions of (1) in two parameters are given by

$$x = x(A, B) = 3(112A^2 - B^2)$$

$$y = y(A, B) = 112A^2 + B^2 + 32AB$$

$$z = z(A, B) = 112A^2 + B^2 + 14AB$$

Properties:

1. $x(1, B) + 3 t_{4,B} \equiv 0 \pmod 2$

2. $y(A, 2) - 112 t_{4,A} - 2G_{32A} \equiv 1 \pmod 2$

3. $y(1,B) - x(1,B) - 4P_B + G_{14B} \equiv 1 \pmod 2$

4. $z(A, 2) - 112t_{4,A} + G_{14A} \equiv 0 \pmod 5$

5. $y(A, 3) - 96 PA - 16t_{4,A} \equiv 1 \pmod 2$

6. $x(1,B) - 2z(1, B) + 5t_{4,B} + G_{14B} \equiv 1 \pmod 2$

7. $y(A, 4) - 112P_A - G_{8A} \equiv 1 \pmod 2$

8. $y(1,B) + z(1,2B) - 5t_{4,B} - G_{30B} \equiv 0 \pmod 5$

Pattern - 3

Assume $z = z(a, b) = a^2+7b^2,\ a, b > 0 \qquad (7)$

Write 16 as $16 = (3 + i\sqrt{7})(3 - i\sqrt{7}) \qquad (8)$

Substituting (7) and (8) in (1), and employing the method of factorization, define

$$(x + i\sqrt{7}\, y) = (3 + i\sqrt{7})\,(a + i\sqrt{7}\, b)^2 \qquad (9)$$

In (9), on equating real and imaginary parts, we get

$$x = x(a, b) = 3a^2 - 21b^2 - 14ab$$

$$y = y(a, b) = a^2 - 7b^2 + 6ab.$$

As our interest centers on finding integer solutions, it is seen that X and Y are integers for suitable choices of a and b. A few illustrations are given below

Case :1 Let $a = 3A$, $b = 3B$

The corresponding solutions of (1) are

$$x = x(A, B) = 9A^2 - 189B^2 - 126AB$$

$$y = y(A, B) = 3A^2 - 63B^2 + 54AB$$

$$z = z(A, B) = 9A^2 + 63B^2$$

Properties

1. $x(A,1) + y(A,1) - 6t_{4,A} + G_{36A} \equiv 1 \pmod 2$

2. $y(1, B) + 63t_{4,B} - G27B \equiv 0 \pmod 2$

3. $x(A, 1) - 9 t_{4,A} + G_{63A} \equiv 0 \pmod 2$

4. $z(1, B) - 63t_{4,B} \equiv 0 \pmod 3$

5. $y(A, 1) + z(A, 1) - 12P_A - G_{21A} - 1 = 0$

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Case 2: Let a = 3A+1, b = 3B+1

The corresponding solutions of (1) are

$x = x(A,B) = 27A^2 - 189B^2 - 24A - 168B - 32$

$y = y(A, B) = 9A^2 - 63B^2 + 24A - 45B.$

$z = z(A, B) = 9A^2 + 63B^2 + 6A + 42B + 8$

Properties

1. $x(A, 1) - 27t_{4,A} + G_{12A} \equiv 0 \pmod 2$
2. $y(1, B) + 45P_B + 18t_{4,B} \equiv 1 \pmod 2$
3. $z(1, B) - 63t_{4,B} + G_{21B} \equiv 0 \pmod 2$
4. $y(2, B) + 45P_B + 18t_{4,B} \equiv 0 \pmod 2$
5. $2y(A,B) - z(A,B) - 9t_{4,A} + 63t_{4,B} - G_{21A} + G_{42B} \equiv 0 \pmod 2$

Pattern: 4

Write (1) as $16z^2 - x^2 = 7y^2$ (12)

Write 7 as $7 = \left(\sqrt{16}+3\right)\left(\sqrt{16}-3\right)$ (13)

Assume $y = y(a,b) = 16a^2 - b^2,\ a, b \neq 0$ (14)

Using (13) and (14) in (12) and employing the method of factorization, define

$$\left(\sqrt{16}z + x\right) = \left(\sqrt{16}+3\right)\left(\sqrt{16}a+b\right)^2 \quad (15)$$

Equating rational and irrational parts in (15), we get

$$\left.\begin{aligned} x &= x(a, b) = 48a^2 + 3b^2 + 32ab \\ z &= z(a, b) = 16a^2 + b^2 + 6ab \end{aligned}\right\} \quad (16)$$

Thus (16) & (14) represent non – zero distinct integral solutions of (1) in two parameters.

Properties:

1. $x(1, B) - t_{8,B} + g_{15B} \equiv 1 \pmod 1$
2. $y(A, 4) - 16t_{4,A} \equiv 0 \pmod 2$
3. $x(A,2) - t_{98,A} + t_{226,A} - 112t_{4,A} \equiv 0 \pmod 2$
4. $x(A,1) + 2y(A,1) - t_{162,A} - t_{6,A} + 2t_{4,A} - g_{56A} \equiv 0 \pmod 2$
5. $x(A, A+1) - 83t_{4,A} - g_{19A} \equiv 0 \pmod 2$
6. $x(4,B) - 3P_B + t_{254,B} - 2g_{63B} \equiv 1 \pmod 2$
7. $y(3, B) + t_{4,B} \equiv 0 \pmod 2$
8. $z(A, 1) - 16t_{4,A} + 2g_{3A} \equiv 0 \pmod 2$
9. $y(A, 1) + z(A, 1) - 32t_{4,A} - 2g_{3A} + 1 = 0$

3. CONCLUSION

In this paper, we have presented four different patterns of non-zero distinct integer solutions to the ternary quadratic Diophantine equation $x^2 + 7y^2 = 16z^2$ representing a cone. To conclude one may search for patterns of non-zero distinct integer solutions satisfying the cone under consideration.

4. REFERENCES

I. Dickson, L.E.,History of Theory of numbers,Vol(2) Chelsea Publishing Company, New York, 1952.

II. Mordell,L.J., Diophantine Equations Academic Press, Londan, 1969.

III. Gopalan.M.A, and Kaliga Rani.,j Observation on Diophantine Equations $y^2 + Dx^2 + z^2$, Impact *J.Sci.Tech.*Vol.(2),No:2, 91-95,2008.

IV. Gopalan. M.A. and Pondichelvi, V., On Ternary Quadratic Equation $x^2 + y^2 = z^2 + 1$, Impact *J.Sci.Tech.* Vol(2), No:2, 55-58, 2008.

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V. Gopalan, M.A., and Vijayasankar. A.,Integral Solutions of Ternary Quadratic Equation $z^2= x^2 + y^2$ proceedings of the International conference of Mathematical methods and computation, Jamal Mohamad College (Autonomous) Tiruchirappalli, India, 68-71, 24-25 July, 2009.