LATTICE POINTS ON THE CONE $x^2 + 9y^2 = 26z^2$

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ABSTRACT

The ternary quadratic homogeneous equation representing cone given by $x^2 + 9y^2 = 26z^2$ *is analyzed for its non-zero distinct integer points on it. Five different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relation between the solutions and special number patterns are presented.*

Key words*: Ternary Quadratic, homogenous cone, integer solutions, special numbers.*
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Notations used:

$t_{m,n}$ *-polygonal number of rank n with size m*
P_n^m*-Pyramidal number of rank n with size m*
g_a *– Gnomonic number of rank a.*
P_n *- Pronic number of rank n.*
SO_n *- Stella octangular number of rank n.*
O_n*-Octahedral number of rank n.*

1. INRODUCTION:

The ternary quadratic Diophantine equations offer an unlimited field for research. This communication concerns with get another interesting ternary quadratic equation representing $x^2 + 9y^2 = 26z^2$ a cone for determining its infinitely many non – zero integral points. Also a few interesting relations among the solutions and special numbers are presented.

2. METHOD OF ANALYSIS:

The ternary quadratic equation to be solved for its non – zero integer solution is

$$x^2 + 9y^2 = 26z^2 \quad (1)$$

Assume $z(a, b) = a^2+9b^2$,where $a, b > 0$ (2)

We illustrate below five different patterns of non–zero distinct integer solutions to (1)

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2.1 Pattern: 1

Write 26 as

$$26 = (5 + i)(5 - i) \quad (3)$$

Substituting (2) & (3) in (1), employing the method of factorization, define

$$(x+3iy)(x-3iy) = (5+i)(5-i)(a+3ib)^2(a-3ib)^2$$

Equating real and imaginary parts, we get

$$x = x(a, b) = 5a^2 - 45b^2 - 6ab \quad (4)$$

$$y = y(a, b) = \frac{1}{3}(a^2 - 9b^2 + 30ab) \quad (5)$$

Thus (2),(4) and (5) represents non- zero distinct integral solutions of (1) in two parameters for suitable a, b. As our interest is on finding integer solutions, we choose a and b suitably so that the values of x, y and z are integers. In what follows the values of a, b ad the corresponding integer solutions are exhibited.

Case:1 Let a = 3A, b = 3B

The corresponding solutions of (1) are

$x = x(a, b) = 45A^2 - 405B^2 - 54AB$

$y = y(a, b) = 3A^2 - 27B^2 + 90AB$

$z = z(a, b) = 9A^2 + 81B^2$

Properties:

1. $x(A, 1) - 45\, t_{4,A} + g_{27A} \equiv 0 \pmod 2$
2. $y(A, A+1) + 2t_{29,A} + g_{38A} \equiv 0 \pmod 2$
3. $x(A, A) - 3z(A, A) - 3P_A + 156\, t_{4,A} = 0$
4. $z(B, B) - 2\, t_{92,B} - g_{44B} = 1$
5. $z(A(A+1), A(A+1)) - 360\, t_{3,A}^2 = 0$

Case: 2 Let a = 3A, b = 3A + 1

The corresponding solutions of (1) are

$x = x(A, A) = 414A^2 + 288A + 45$

$y = y(A, A) = 66A^2 + 12A - 3$

$z = z(A, A) = 90A^2 + 54A + 9$

Properties:

1. $x(A(A+1),1) - 414P_A^2 - 288P_A \equiv 0 \pmod 5$
2. $z(A,A) - 90t_{4,A} - g_{27A} \equiv 0 \pmod 2$
3. $y(A^2(A+1),1) - 132(P_A^5)^2 - 24P_A^5 \equiv 0 \pmod 3$
4. $z(A(2^A+1),1) - 270O_A^2 - 162O_A \equiv 0 \pmod 3$
5. $x(A(2A^2-1),1) - 414SO_A^2 - 288SO_A + 45 \equiv 0 \pmod 5$

Case: 3 Let a = 3A, b = B

The corresponding solutions of (1) are

$x = x(a, b) = 45A^2 - 45B^2 - 18A$

$y = y(a, b) = 3A^2 - 3B^2 + 30AB$

$z = z(a, b) = 9A^2 + 9B^2$

Properties:

1. $x(A, A+1) - 15y(A, A+1) + 48P_A = 0$
2. $3y(A+1, A)\, z(A+1, A) + 18t_{4,A} - 30P_A = 0$
3. $x(A(A+1),1) - 45P_A^2 + 18P_A \equiv 0 \pmod 2$
4. $12\{y(A, A) - 30\, t_{4,A}\}$ a nasty number
5. $24\{z(A,B) - 9\, t_{4,A} - 9\, t_{4,B}\}$ a nasty number

2.2 Pattern: 2

Instead of (3), write 26 as

$$26 = (1 + 5i)(1 - 5i) \quad (6)$$

Following the procedure presented as in pattern:1, the corresponding values of and y are

$$x = x(a, b) = a^2 - 9b^2 - 30ab \quad (7)$$

$$y = y(a, b) = \frac{1}{3}(5a^2 - 45b^2 + 6ab) \quad (8)$$

Thus (2), (7) and (8) represents non- zero distinct integral solutions of (1) in two parameters.

Case: 1 Let a = 3A, b = 3B

The corresponding solutions of (1) are

$x = x(a, b) = 9A^2 - 81B^2 - 270AB$

$y = y(a, b) = 15A^2 - 135B^2 + 18AB$

$z = z(a, b) = 9A^2 + 81B^2$

Properties:

1. $x(A^2(A+1),1) - 18(P_A^5)^2 + 540P_A^5 \equiv 0 \pmod 3$
2. $x(A, B) + 15y(A, B) - 234\, t_{4,A} + 2108\, t_{4,B} = 0$
3. $z(A(A+1), B(B+1)) - 9P_A^2 - 81P_B^2 = 0$

4. 12{z (A, A) – 90 $t_{4, A}$} a nasty number
5. x $(n(2n^2–1), 1)+261SO_n \equiv 0 \pmod 3$

2.3 Pattern: 3

(1) is written in the form of ratio as

$$\frac{x+z}{5z+3y} = \frac{5z-3y}{x-z} = \frac{A}{B}, \quad B \neq 0,$$

which is equivalent to the system of equations

$B(x + z) – A (5z + 3y) = 0$ (9)

$B(5z-3y) - A(x – z) = 0$ (10)

Applying the method of cross- multiplication the integer solutions of (1) are given by

$x = x (A, B) = 3A^2 – 3B^2 + 30AB$ (11)

$y = y (A, B) = 5A^2 – 5B^2 – 2AB$ (12)

$z = z (A, B) = 3A^2 + 3B^2$, (13)

which represents non – zero distinct integral of (1) in two parameters

Properties:

1. $x(A(A+1),B(B+1))–5y(A(A+1),B(B+1)) – 72 P_A^2 + 78 P_B^2 = 0.$
2. $y (A,1) – 5 t_{4, A} + g_A \equiv 0 \pmod 2$
3. $z (A, B) – 3 t_{4,A} -3 t_{4, B} = 0$
4. $x (A, 1) – 30 P_A + 27 t_{4, A} \equiv 0 \pmod 3$
5. 24{z(A, B)- $3t_{4,A}$- $3t_{4,B}$ } a nasty number.

2.4 Pattern: 4

(1) is written as

$26z^2 – 9y^2 = x^2 = x^2 * 1$ (14)

Assume $x (a, b) = 26a^2 – 9b^2$ (15)

Write(1)as1$= (\sqrt{26} + 5)(\sqrt{26} - 5)$ (16)

Substituting (15) and (16) in (14) and applying the method of factorization, define

$(\sqrt{26}\ z + 3y) = (\sqrt{26}\, a + 3b)^2(\sqrt{26} - 5)$

Equating rational and irrational parts, we have

$y = y (a, b) = \frac{1}{3}\{130a^2 + 45b^2 – 156ab\}$ (17)

$z = z (a, b) = 26a^2 + 9b^2 – 30ab$ (18)

Case: 1 Let a = 3A, b = 3B

The corresponding solutions of (1) are

$x = x (a, b) = 234A^2 – 81B^2$

$y = y (a, b) = 390A^2 + 135B^2 – 468AB$

$z = z (a, b) = 234A^2 + 81B^2 – 270AB$

Properties:

1. $x (A(A+1), 1) – 234 P_A^2 \equiv 0 \pmod 3$
2. $x (A, 1)–z(A, 1) + g_{135A} \equiv 1 \pmod 2$
3. $x (A, B) – 234 t_{4, A} + 81 t_{4, B} =0$
4. $z (A, 1) – 2t_{236, A} + g_{19A} + 1 = 0$
5. $x (A, A) – 153 t_{4, A} = 0.$

Case: 2 Let a = 3A, b = 3B+1

The corresponding solutions of (1) are

$x = x (a, b) = 234A^2 – 81B^2 – 54B – 9$

$y = y (a, b) = 390A^2+135B^2-468A+90B–1404AB + 15$

$z = z (a, b) = 234A^2 + 81B^2 – 90A + 54B – 270AB + 9.$

Properties:

1. $x (A(A+1), 1) – 234 P_A^2 + 144 = 0$
2. $y (A, A) + 879 t_{4, A} + g_{189 A} \equiv 0 \pmod 2$
3. $z ((B, 1) – 2t_{236, B} + g_{64 B} + 143 = 0$
4. $x (A, A) – 153 t_{4, A} + g_{27A} \equiv 0 \pmod 2$
5. $y (B, 1) – 390 t_{4, B} + g_{936B} \equiv 1 \pmod 2$

2.5 Pattern: 5

Assume $x (a, b) = 26a^2 – 9b^2$ (19)

Substituting (19) in (1) and applying the method of factorization, define

$(\sqrt{26}\ a + 3b)^2 = (\sqrt{26}\ z + 3y)$

Equating rational and irrational, we get

$y = \frac{1}{3}\{26a^2 + 9b^2\}$.(20)

$z = 6ab$ (21)

As our interest is on finding integer solutions, We choose a and b suitably so that the values of x, y and z are in integers. In what follows the values of a, b and the corresponding integer solutions are exhibited.

Case:1 Let a = 3A, b = 3B

The corresponding solutions of (1) are

$x = x (a, b) = 234A^2 – 81B^2$

$y = y (a, b) = 78A^2 + 27B^2$

$z = z (a, b) = 54AB$

Properties:

1. $x (A(A+1), 1) – 234 P_A^2 \equiv 0 \pmod 3$
2. $x (A, B) – 234 t_{4, A} + 81 t_{4, B} = 0$
3. $z (A, A+1) – 54 P_A = 0$

4. $y(A, 1) - 78P_A + g_{39A} \equiv 0 \pmod 2$
5. $y(A, B) - 2t_{80, A} - g_{38A} - 27\, t_{4, B} = 1$

Case:2 Let $a = 3A$, $b = B$

The corresponding solutions of (1) are

$x = x(a, b) = 234A^2 - 9B^2$

$y = y(a, b) = 78A^2 + 3B^2$

$z = z(a, b) = 18\, AB$

Properties:

1. $z(A(A+1), 1) - 18\, P_A = 0$
2. $y(A(A+1), B(B+1)) - 78\, P_A^2 - 3P_B^2 = 0.$
3. $x(A,A) - 225\, t_{4, A} = 0$
4. $z(A,A) - 18\, t_{4, A} = 0$
5. $y(A, B) - 78t_{4, A} - 3t_{4, B} = 0$

Case: 3 Let $a = 3A$, $b = 3B + 1$

The corresponding solutions of (1) are

$x = x(a, b) = 234A^2 - 81B^2 - 54B - 9$

$y = y(a, b) = 78A^2 + 27B^2 + 18B + 3$

$z = z(a, b) = 18A + 54AB$

Properties:

1. $x(A, A) - 153\, t_{4, A} + 2\, g_{27A} \equiv 0 \pmod{10}$
2. $y(B(B+1), 1) - 78\, P_B^2 \equiv 0 \pmod{48}$
3. $z(A(A + 1)) - g_{9A} - 54\, P_A = 1$
4. $y(A, B) - 78\, t_{4, A} - 27\, t_{4, B} - g_{9B} \equiv 0 \pmod 4$
5. $z(A, A) - 2t_{56, A} - g_{35A} = 1$

3. CONCLUSION:

In this work, the ternary quadratic Diophantine equations refereeing a conies is analysed for its non - zero distinct integral points. A few interesting properties between the solutions and special numbers are presented. To conclude, one may search for other patterns of solutions and their corresponding properties for the cone under consideration

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